ON ISOMETRIC AND MINIMAL ISOMETRIC EMBEDDINGS

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Abstract. In this paper we study critial isometric and minimal isometric embeddings of classes of Riemannian metrics which we call *quasi-κ-curved metrics*. Quasi-κ-curved metrics generalize the metrics of space forms. We construct explicit examples and prove results about existence and rigidity.

Introduction

Definition: Let $(M^n, \tilde{g})$ be a Riemannian manifold. We will say $\tilde{g}$ is a *quasi-κ-curved metric* if there exists a smooth positive definite quadratic form Q on M such that for all $x \in M$

$$R_x = -\gamma(Q_x, Q_x) + (\kappa + 1)\gamma(\tilde{g}_x, \tilde{g}_x) \tag{1}$$

where $\gamma : S^2T^* \to S^2(\Lambda^2 T^*)$ denotes the algebraic Gauss mapping and R_x the Riemann curvature tensor. (See §1. for more details.)

Quasi-κ-curved metrics are a generalization of *quasi-hyperbolic metrics* defined in [BBG], which correspond to $\kappa = -1$. We will also refer to the case $\kappa = 0$ as *quasi-flat metrics*. We will assume that $n \geq 3$. When $n = 3$, the quasi-κ-curved condition is an open condition on the metric, and thus in this case the class of metrics we study is quite general. The condition is stronger in higher dimensions.

In this paper we study local isometric embeddings and minimal isometric embeddings of quasi-κ-curved manifolds. Before giving our results, it will be useful to review some of what is known:

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Local isometric embeddings

Given a Riemannian manifold $(M^n, \tilde{g})$, one may ask if it admits a local isometric embedding into a Euclidean space $\mathbb{R}^{n+r}$ or more generally a space form $X(\epsilon)^{n+r}$ of constant sectional curvature ϵ. If M is more positively curved than X, one expects to have local isometric embeddings (e.g. the embedding of S^n into $\mathbb{R}^{n+1}$). We will be concerned with the case M is less positively curved than X.

The critical codimension for the isometric embedding problem is $r = \binom{n}{2}$. The Cartan-Janet theorem states that local isometric embeddings of analytic Riemannian n-folds into $\mathbb{R}^{n+\binom{n}{2}}$ exist and depend locally on a choice of n arbitrary functions of $n-1$ variables. (These "dimension counts" come from the Cartan-Kähler Theorem [Car2].) In this paper, we will be interested in the overdetermined case $r < \binom{n}{2}$.

In the most overdetermined case ($r = 1$), Thomas [T] observed that the Codazzi equations of a hypersurface with non-degenerate second fundamental form are consequences of the Gauss equations when $\dim(M) \geq 4$. Thus codimension one questions reduce to questions in multi-linear algebra. (See [CO] for a clear exposition.)

For $r \leq n$, Cartan [Car1] studied the isometric immersions of a flat space into a Euclidean space, showing that if $r \leq n-1$ there is no local isometric embedding (other than the totally geodesic one) and that when $r = n$ such embeddings depend on $\binom{n}{2}$ functions of 2 variables. In the course of his proof, Cartan proved a basic theorem about exteriorly orthogonal symmetric bilinear forms (see e.g. [Spivak], V.11.5). Using a reduction to the flat case (see §1), Cartan went on to show that hyperbolic space H^n admits no local isometric embedding into $\mathbb{R}^{2n-2}$ but admits local isometric embeddings into $\mathbb{R}^{2n-1}$. Moreover, these embeddings depend on a choice of $n^2 - n$ functions of one variable, which is the most possible for local isometric embeddings (with nondegenerate second fundamental form) for any n-fold into $\mathbb{R}^{2n-1}$.

Cartan's work implies that for such isometric embeddings of hyperbolic space there is, at each point, an orthonormal basis of the tangent space under which the second fundamental form II is diagonalized with respect to the metric. Thus, for such embeddings there are principal tangent directions and princial normal vectors, in analogy with the case of hypersurfaces.

Quasi-κ-curved metrics share the property that, for immersions to $X(\kappa+1)$ in the critical codimension, the tangent space admits a basis for which the second fundamental form is diagonalized, but the basis is Q-orthonormal instead of $\tilde{g}$ orthonormal. We will refer to these vectors as the *principal tangents*, and the corresponding values of II as *principal normal vectors*.

Cartan's results for hyperbolic space were generalized by Berger, Bryant and Griffiths [BBG] to quasi-hyperbolic metrics. They showed that no quasi-hyperbolic metric admits a local isometric embedding to Euclidean space when $r < n-1$, and characterized the quasi-hyperbolic metrics that admit local isometric embeddings in the critical case $r = n-1$.

To understand this characterization, note that the Riemann curvature tensor for a quasi-hyperbolic metric is like that of a hypersurface in Euclidean space but the sign is wrong. If instead one considers spacelike hypersurfaces in Lorentz space, one obtains quasi-hyperbolic metrics where $Q = II$. In [BBG] they assert that *all* quasi-hyperbolic metrics arise in this way (at least when $n > 3$).

The sub-class of quasi-hyperbolic metrics satisfying the integrability conditions to admit a critical isometric embedding are precisely the spacelike hypersurfaces satisfying the condition

$$\nabla II_{\mathbb{L}} = L \cdot II_{\mathbb{L}}, \tag{2}$$

where $II_{\mathbb{L}}$ denotes the second fundamental form of the hypersurface embedding, ∇ is the connection form of the metric $\tilde{g}$ and $L \in \Omega^1(M)$ is some linear form. (This is in fact an intrinsic condition because $II_{\mathbb{L}} = Q$.) A result in projective geometry ([GH], B.16), which specializes to affine geometry, shows that (2) occurs if and only if M^n is embedded as a patch of a quadric hypersurface in affine space. In [BBG] they also assert that in the $n = 3$ case all quasi-hyperbolic 3-folds satisfying the integrability conditions occur as quadric hypersurfaces in $\mathbb{L}^4$. Metrics satisfying (2) admit local isometric embeddings into $\mathbb{R}^{2n-1}$ which depend on a choice of $n^2 - n$ functions of one variable, as in the case of hyperbolic space.

The [BBG] result generalizes to quasi-κ-curved metrics:

Theorem A. *Let (M^n, g), $n \geq 3$, be a quasi-κ-curved Riemannian manifold. Let $X^{2n-1}(\kappa + 1)$ be a space form with constant sectional curvature $\kappa + 1$. Then there exist local isometric embeddings $M^n \hookrightarrow X^{2n-1}(\kappa + 1)$, with local solutions depending on $n^2 - n$ functions of one variable, if and only if ∇Q is a symmetric cubic form on M and*

$$\nabla Q = L \cdot Q$$

for some linear form $L \in \Omega^1(M)$.

To produce quasi-hyperbolic metrics, it was natural to look at hypersurfaces in $\mathbb{L}^{n+1}$. To study quasi-κ-curved metrics, it is natural to look for codimension two spacelike submanifolds of $\mathbb{L}^{n+2}$ having a spacelike section σ of the unit normal bundle such that $\sigma \lrcorner II = \sqrt{\kappa + 1} g$. Such M are quasi-κ-curved with quadratic form $Q = \sigma^{\perp} \lrcorner II$, where $\sigma^{\perp}$ is the timelike section of length -1 normal to σ. (We do not know whether or not all quasi-κ-curved metrics arise in this way.) Using methods developed in [Lan], we first show any such spacelike submanifold is in fact a hypersurface inside a sphere:

Lemma. *Let $M^n \subset \mathbb{L}^{n+2}$ be a spacelike submanifold of Lorentz space. Suppose there exists a section σ of the unit normal bundle of M, such that*

$$\sigma \lrcorner II = cg,$$

where c is some constant and g is the induced metric. Then M is conguent to a submanifold of the Lorentzian sphere of radius $1/c$.

Remark: The lemma is valid in more general contexts which will be suggested in the proof.

Corollary. *Spacelike hypersurfaces of the Lorentzian sphere of radius* $\frac{1}{\sqrt{\kappa+1}}$ *in* $\mathbb{L}^{n+2}$ *are quasi-*κ*-curved.*

Let $x^0, x^1, \cdots, x^{n+1}$ be coordinates on $\mathbb{L}^{n+2}$ such that

$$< x, x >= -(x^0)^2 + (x^1)^2 + \cdots + (x^{n+1})^2.$$

Then the sphere our computations will produce has the equation:

$$-(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 + (x^{n+1} - \frac{1}{2\sqrt{\kappa+1}})^2 = \frac{1}{\kappa+1} \tag{3}$$

with $\sigma_{(0,\ldots,0)} = \partial/\partial x^{n+1}$ and $T^*_{(0,\ldots,0)}M = \{dx^i\}$, $1 \le i \le n$. Note that for the quasi-hyperbolic case, (3) specializes to the linear subspace $x^{n+1} = 0$; the sphere of radius infinity.

We also classify those hypersurfaces that satisfy the isometric embedding criteria, depending on the multiplicity of the eigenvalues of Q with respect to g.

Theorem B. *Let* $M^n \subset \mathbb{L}^{n+2}$ *be quasi-*κ*-curved with nondegenerate quadratic form* Q*.*

Case 1. If, on an open set of M*, there exists an eigenvalue of* Q *of multiplicity one, then* $\nabla Q = LQ$ *for some linear form* L *if and only if* M *is the intersection of (3) and the quadric*

$$0 = x^0 - q_{ij}x^i x^j - \lambda_j x^0 x^j - b(x^0)^2 - (\kappa+1)x^0 x^{n+1} \tag{4}$$

where q_{ij}, λ_j *are respectively the coefficients of* Q *and* L *at* $(0, \ldots, 0)$ *with respect to the orthonormal basis* dx^i*, and* b *is a constant.*

Case 2. If, on an open set of M*, there are no eigenvalues of muliplicity one, then* $\nabla Q = LQ$ *if and only if* $\nabla Q = 0$*, in which case* M *is a product of space forms and is the intersection of (3) with the quadric*

$$0 = x^0 - q_{ij}x^i x^j - b(x^0)^2 - (\kappa+1)x^0 x^{n+1} \tag{5}$$

where q_{ij} *are the coefficients of* Q *at* $(0, \ldots, 0)$ *with respect to the orthonormal basis* dx^i*, and* b *is a constant.*

Minimal isometric embeddings

Thanks to the work of Calabi, there is a reasonable understanding of the Riemannian metrics of minimal surfaces. However in higher dimensions, almost nothing is known beyond algebraic restrictions coming from the Gauss equations and restrictions coming from the isometric embedding alone.

Calabi showed that any surface $M^2 \subset \mathbb{R}^n$ that is minimally and isometrically embedded must arise as a projection from some Hermitian isometric embedding of M into some $\mathbb{C}^m$, where $m \le n \le 2m$ (see [Law]). From this description, it

follows that any such M always admits some number of constants' worth (again, in the Cartan-Kähler language) of noncongruent minimal isometric embeddings, but never any functions' worth. For example, non-planar minimal surfaces in $\mathbb{R}^3$ always admit a one parameter family of minimal isometric deformations, the most famous of which is the family connecting catenoids and helicoids. One may wish to contrast this situation with the isometric embedding problem for surfaces in $\mathbb{R}^3$, without the requirement of minimality, where, as stated above, a generic metric admits two functions of one variable's worth of noncongruent isometric embeddings ([Car1]). (In fact, one can pose a Cauchy problem with a space curve as initial data and realize these two functions as the curvature and torsion of the curve.)

It follows from Calabi's work that no patch of the hyperbolic plane admits a minimal isometric embedding to any finite dimensional Euclidean space. On the other hand, Calabi showed that all finite dimensional hyperbolic spaces admit a minimal isometric embedding into a Hilbert space ([Cal]).

Moore [M] proved that the only ways in which an n-dimensional space form M^n can be locally isometrically embedded as a minimal submanifold in a space form $X^N(\epsilon)$ with $N \leq 2n-1$ are if the image is totally geodesic, or M^n is flat and its image is a piece of the Clifford torus T^n in S^{2n-1}. In particular, H^n admits no local minimal isometric embeddings into $\mathbb{R}^{2n-1}$.

Define the *rank* of an embedding as the dimension of the image of the second fundamental form as a linear map

$$\mathrm{II} : S^2 T_x M \longrightarrow N_x M.$$

In other words, the dimension of the second osculating space of M at a point is n plus the rank. We obtain the following extensions of Moore's theorem:

Theorem C.

1. If $(M^n, \tilde{g})$ is a quasi-κ-curved manifold and Q is $\tilde{g}$-parallel, then M does not admit any local embedding of rank less than n into $X^N(\epsilon)$, with $\epsilon < \kappa$, which is isometric and minimal, or which is isometric with parallel mean curvature vector, except when M is flat and its image is a piece of the Clifford torus.

2. If an n-dimensional space form $M^n(\kappa)$ is locally isometrically embedded as a minimal submanifold of constant rank in an N-dimensional space form $X^N(\epsilon)$, then either M is totally geodesic, or the rank is at least $n-1$. If the rank is exactly $n-1$, then either M is totally geodesic or M is flat and its image is a piece of the Clifford torus T^n in S^{2n-1}. (In particular, H^n admits no local minimal isometric embeddings of constant rank $n-1$ into $\mathbb{R}^N$.)

We also obtain the following rigidity theorem:

Theorem D. *Let M^n be a quasi-κ-curved Riemannian manifold. Let ℓ be the dimension of the principal orbits of the identity component of the isotropy group of M, at a generic point $p \in M$, acting on the Q-orthonormal frames of T_pM. Then the minimal isometric embeddings, and more generally the isometric embeddings*

with parallel mean curvature vector, of M into a space form $X^{2n-1}(\kappa+1)$ depend, up to rigid motions, on at most a choice of $\binom{n}{2}-\ell$ constants.

(Note that, when γ is followed by the Ricci trace, the result is a map of full rank on the space of quadratic forms at a positive definite Q. Hence the identity component of the isometry group of M preserves Q.)

The minimality condition imposes additional integrability conditions on the metric; these take the form of an overdetermined set of polynomials involving Q, the curvature tensor, and their covariant derivatives. Because of Theorem C, we know these additional conditions are non-trivial, and so we obtain

Theorem E. *A non-empty Zariski-open subset of the space of quasi-κ-curved manifolds M^n that admit a local isometric embedding into a space form $X^{2n-1}(\kappa+1)$ does not admit any minimal isometric embedding.*

Outline of the paper

In §1 we review the algebraic form of the Gauss equations of a submanifold of a space form and explain the quasi-κ-curved condition in more detail. In §2 we set up the isometric embedding problem following [BBG] and prove Theorem A. In §3 we describe what happens to the system when we add the minimality condition, and we prove Theorems C and D in the case of minmality. In §4 these results are extended to the case of parallel mean curvature vector. In §5 we construct quasi-κ-curved n-folds as submanifolds of spheres in $\mathbb{L}^{n+2}$, proving the Lemma and Theorem B.

§1. The Gauss equations

The class of Riemannian metrics we will be dealing with have a special property that is best described in terms of the algebraic form of the Gauss equations of a submanifold of a space form.

Throughout this paper, let $V=\mathbb{R}^n$ and $W=\mathbb{R}^r$; we endow W with the standard inner product.

Let K be the kernel of the skew-symmetrization map:

$$0 \longrightarrow K \longrightarrow \Lambda^2V^*\otimes\Lambda^2V^* \longrightarrow V^*\otimes\Lambda^3V^*.$$

Note that actually $K\subset S^2(\Lambda^2V^*)$. (This is often called the first Bianchi identity.) K is the space of tensors with the symmetries of the Riemann curvature tensor.

Let

$$\gamma : S^2V^*\otimes W\to K$$

be the $Gl(n)\times O(r)$-equivariant quadratic map defined as follows: in terms of an arbitrary basis $\{e^i\}$ for V^* and an orthonormal basis $\{w_\mu\}$ for W, γ is given by

$$h^\mu_{ij}e^ie^j\otimes w_\mu \mapsto \Sigma_\mu(h^\mu_{ik}h^\mu_{jl}-h^\mu_{il}h^\mu_{jk})(e^i\wedge e^j)\otimes(e^k\wedge e^l).$$

For a submanifold $M^n \subset X^{n+r}(\epsilon)$ with the induced metric, we will have $V = T_xM$, $W = N_xM$ (the fibre of the normal bundle at x), and

$$R = \gamma(II, II) + \epsilon\gamma(g, g) \tag{6}$$

where here and in what follows, to apply γ to an element of S^2V^*, just take $W = \mathbb{R}$.

As mentioned in the introduction, Cartan realized that one could study isometric embeddings of hyperbolic space via the isometric embeddings of flat space. For, the curvature tensor R_0 of hyperbolic space has the property that it is minus the image of the metric $g_0 \in S^2V^*$ under the map γ:

$$R_0 = -\gamma(g_0, g_0). \tag{7}$$

Letting $\hat{W} = W \oplus \mathbb{R}$ and $\widehat{II} = II \oplus g_0$, in the case of hyperbolic space, one obtains

$$0 = \gamma(\widehat{II}, \widehat{II}), \tag{8}$$

the Gauss equations for an isometric embedding of a flat metric into $\mathbb{R}^{n+r+1}$.

The Gauss equations for embedding a quasi-κ-curved metric $\tilde{g}$ into a space form of curvature $\kappa + 1$ have the form

$$-\gamma(Q, Q) + (\kappa + 1)\gamma(\tilde{g}, \tilde{g}) = \gamma(II, II) + (\kappa + 1)\gamma(\tilde{g}, \tilde{g})$$

When one defines $\widehat{II} = II \oplus Q$, once again the Gauss equations take the form (8).

§2. Moving frames and the isometric embedding system

In this section we summarize what we will need from [BBG].

Let $(M, \tilde{g})$ be a Riemannian manifold. Let $\tilde{\pi} : \tilde{\mathcal{F}} \to M$ denote the bundle of all frames of TM, i.e., the fiber over a point $x \in M$ is the set of all bases $\tilde{e}_1, \ldots, \tilde{e}_n$ of T_xM. On $\tilde{\mathcal{F}}$, write $dx = \tilde{\omega}^i \tilde{e}_i$ where the one-forms $\tilde{\omega}^i$ are semi-basic to the projection $\tilde{\pi}$. Let $\tilde{g}_{ij} = \tilde{g}(\tilde{e}_i, \tilde{e}_j)$. On $\tilde{\mathcal{F}}$ we have structure equations

$$\begin{aligned} d\tilde{g}_{ij} &= \tilde{g}_{ik}\tilde{\omega}^k_j + \tilde{g}_{kj}\tilde{\omega}^k_i \\ d\tilde{\omega}^i &= -\tilde{\omega}^i_j \wedge \tilde{\omega}^j \\ d\tilde{\omega}^i_j + \tilde{\omega}^i_k \wedge \tilde{\omega}^k_j &= \tilde{\Omega}^i_j, \end{aligned}$$

where the forms $\tilde{\omega}^i_j$ are connection forms for the Levi-Civita connection associated to $\tilde{g}$ and the forms $\tilde{\Omega}^i_j$ are the curvature two-forms for this connection.

We will set up an isometric embedding system for quasi-κ-curved metrics following [BBG]. Let $\epsilon = \kappa + 1$ and let $X(\epsilon)^{n+r}$ denote the space form of constant

sectional curvature ϵ. Let $\mathcal{F} \to X(\epsilon)^{n+r}$ denoted the frame bundle adapted such that for $x \in X$ the fibre of $\mathcal{F}$ consists of bases (e_i, e_μ) for $T_x X$, such that

$$e_i \cdot e_\mu = 0, \qquad e_\mu \cdot e_\nu = \delta_{\mu\nu}$$

(We will use index ranges $1 \leq i,j,k \leq n$ and $n+1 \leq \mu,\nu \leq n+r$. The "$\cdot$" is the standard inner product.) Let $g_{ij} = e_i \cdot e_j$ and let g^{ij} be the components of g^{-1}. If we treat $f = (x, e_i, e_\mu)$ as a matrix-valued function on $\mathcal{F}$, and write $df = f\Omega$, then Ω is a matrix-valued 1-form, the *Maurer-Cartan form*. We will denote the entries of Ω as follows:

$$\Omega = \begin{pmatrix} 0 & -\epsilon\omega^j & -\epsilon\omega^\nu \\ \omega^i & \omega^i_j & \omega^i_\nu \\ \omega^\mu & \omega^\mu_j & \omega^\mu_\nu \end{pmatrix}$$

where we have symmetries

$$\omega^\mu_\nu = -\omega^\nu_\mu, \qquad \omega^k_\nu = -g^{ki}\omega^\nu_i.$$

It follows from the definition of Ω that

$$dg_{ij} = g_{ik}\omega^k_j + g_{jk}\omega^k_i.$$

The *Maurer-Cartan equation*, $d\Omega = -\Omega \wedge \Omega$ enables us to compute the exterior derivatives of the components of Ω.

On the submanifold $\Sigma \subset \mathcal{F} \times \tilde{\mathcal{F}} \times (S^2V^* \otimes W)$ defined by the equations $\tilde{g}_{ij} - g_{ij} = 0$, define the Pfaffian system

$$I^{std} = \{\omega^i - \tilde{\omega}^i, \omega^\mu, \omega^i_j - \tilde{\omega}^i_j, \omega^\mu_i - h^\mu_{ij}\omega^j\}.$$

with independence condition $\tilde{\omega}^1 \wedge \ldots \wedge \tilde{\omega}^n \neq 0$. Integral n-manifolds of this system are graphs, on the level of frames, of isometric embeddings of $(M, \tilde{g})$. By differentiating the last set of forms in I^{std}, one sees that integral manifolds can only lie in the subset $\Sigma' \subset \Sigma$ where the Gauss equations (6) are satisfied. Also, note that this EDS is invariant under the group $GL(n) \times O(r)$, acting by orthonormal changes of basis among the e_μ and arbitrary but simultaneous changes of basis among $\tilde{e}_1, \ldots, \tilde{e}_n$ and among $e_1, \ldots, e_n$.

Henceforth we will assume that $r = n-1$ and that the metric $\tilde{g}$ is quasi-κ curved with respect to a positive definite quadratic form Q.

Now the Gauss equations take the form (8). By an application of Cartan's theorem on exteriorly orthogonal forms, there exists at each point a Q-orthonormal basis for TM such that:

$$h^\mu_{ij} = \delta_{ij} b^\mu_i, \tag{9}$$

and the n vectors $b_i = (b^\mu_i) \in W$ satisfy

$$b_i \cdot b_j = -1 \text{ for } i \neq j. \tag{10}$$

(See [BBG], §§4.2ff for details.) Note that such a basis is unique up to permutation.

Let $\pi_Q : \tilde{\mathcal{F}}_Q \to M$ denote the bundle of Q-orthonormal frames of TM, to which we restrict all the forms defined on $\tilde{\mathcal{F}}$.

Let $\mathcal{W}$ be the smooth submanifold of $V^* \otimes W$ defined by (10). On the submanifold $\Sigma'' \subset \tilde{\mathcal{F}}_Q \times \mathcal{F} \times \mathcal{W}$ defined by the equations $\tilde{g}_{ij} - g_{ij} = 0$, define the Pfaffian system

$$I = \{\tilde{\omega}^i - \omega^i, \omega^\mu, \tilde{\omega}^i_j - \omega^i_j, \omega^\mu_i - b^\mu_i \omega^i\}. \tag{11}$$

(There is no sum on i in the last group.)

We will need to compute the derivatives of these forms modulo I. For all but the last group of forms in (11), the exterior derivatives are zero modulo I. The exterior derivatives of the forms in the last group provide the coefficients of the covariant derivative of the second fundamental form, ∇II. Following [BBG], to facilitate computations we write

$$db^\mu_i - \omega^\mu_\nu b^\nu_i = \sum_j (b^\mu_i - b^\mu_j)\pi_{ij}. \tag{12}$$

(12) defines the one forms π_{ij}. (12) is possible because it follows from (10) that for any i the vectors $b_i - b_j$ are a basis for W. By convention, $\pi_{ii} = 0$ for any i.

It is also convenient to keep track of the difference between the connections defined by g and Q. To this end, write

$$\tilde{\omega}^i_j = \tilde{\mu}^i_j + \tilde{\nu}^i_j, \tag{13}$$

where $\tilde{\mu}^i_j = \tilde{\mu}^j_i$ and $\tilde{\nu}^i_j = -\tilde{\nu}^j_i$. The $\tilde{\mu}^i_j$ measure the difference between the Levi-Civita connections of $\tilde{g}$ and Q. Accordingly, there are tensor components $\tilde{\mu}^i_{jk}$ such that $\tilde{\mu}^i_j = \sum_k \tilde{\mu}^i_{jk}\tilde{\omega}^k$ on $\tilde{\mathcal{F}}$. The contraction

$$\nabla Q = (Q_{il}\tilde{\mu}^l_{jk} + Q_{jl}\tilde{\mu}^l_{ik})\tilde{\omega}^i\tilde{\omega}^j \otimes \tilde{\omega}^k$$

is the covariant derivative of Q with respect to the connection of $\tilde{g}$. ∇Q, denoted by III in [BBG], will play a role in what follows.

There is a (relatively harmless) error in [BBG] in computing $d(\omega^\mu_i - b^\mu_i \omega^i)$ modulo I. (See the equation above (4.33) in that paper.) We are grateful to Robert Bryant for indicating how to make a correction. For the sake of completeness, we include it here:

$$d(\omega^\mu_i - b^\mu_i\omega^i) = -\sum_j \omega^\mu_j \wedge \omega^j_i - \sum_\nu \omega^\mu_\nu \wedge \omega^\nu_i - db^\mu_i \wedge \omega^i + \sum_j b^\mu_i \omega^i_j \wedge \omega^j$$

$$\equiv -(db^\mu_i + \omega^\mu_\nu b^\nu_i) \wedge \omega^i + \sum_j (b^\mu_i \tilde{\omega}^i_j \wedge \omega^j + b^\mu_j \tilde{\omega}^j_i \wedge \omega^j) \bmod I \tag{14}$$

$$\equiv \sum_j (b^\mu_i - b^\mu_j)\left(-\pi_{ij} \wedge \omega^i + \tilde{\omega}^i_j \wedge \omega^j\right) + 2\sum_j b^\mu_j \tilde{\mu}^i_j \wedge \tilde{\omega}^j \bmod I \tag{15}$$

(Note that there is no sum on the index i.) Define the functions

$$B_i = b_i \cdot b_i + 1$$

on $\mathcal{W}$; then

$$b_i \cdot b_j = -1 + \delta_{ij} B_i \tag{16}$$

for any i, j. Dotting (15) with b_k, $k \neq i$, shows that the two-forms in the system are

$$B_k(\pi_{ik} \wedge \tilde{\omega}^i - \tilde{\nu}^i_k \wedge \tilde{\omega}^k + \tilde{\mu}^i_k \wedge \tilde{\omega}^k) - 2 \sum_j \tilde{\mu}^i_j \wedge \tilde{\omega}^j, \qquad i \neq k \tag{17}$$

Compared with [BBG], our formula has an extra term (the one with the sum over j) at the end. Nevertheless we will derive the same integrability condition:

$$\tilde{\mu}^i_j \equiv 0 \bmod \tilde{\omega}^i, \tilde{\omega}^j \text{ for } i \neq j. \tag{18}$$

This condition places additional restrictions on the metric $\tilde{g}$. In fact, there must be contravariant tensor components λ_k on $\tilde{\mathcal{F}}$ such that

$$\tilde{\mu}^i_j = \lambda_i \tilde{\omega}^j + \lambda_j \tilde{\omega}^i + \delta^i_j \sum_k \lambda_k \tilde{\omega}^k. \tag{19}$$

(See pp. 866-867 in [BBG] for details.) This in turn implies that the extra term in (17) is zero, and the rest of the argument in [BBG], from (4.35) onwards, goes through with $\tilde{\mu}^i_j$ replaced by $-\tilde{\mu}^i_j$.

To derive (18) from our correction, first note that (10) implies that

$$\sum_i \frac{1}{B_i} = 1. \tag{20}$$

For any i and j, let

$$\Theta^i_j = \tilde{\omega}^i_j \wedge \tilde{\omega}^j \wedge \tilde{\omega}^i.$$

Then wedging the two-form (14) with $\tilde{\omega}^i$ gives $\sum_j (b_i \Theta^i_j - b_j \Theta^j_i)$. Dotting with b_k for $k \neq i$ gives the three-form $-B_k \Theta^k_i + \sum_j (\Theta^j_i - \Theta^i_j)$. Thus, on any integral manifold of I,

$$\Theta^k_i = \sum_j (\Theta^j_i - \Theta^i_j)/B_k. \tag{21}$$

Summing over $k \neq i$, using (20) and solving gives

$$\sum_j \Theta^i_j = (1 + B_i) \sum_j \Theta^j_i.$$

Substituting this into (21) and summing both sides over k gives

$$\sum_j \Theta^j_i = (1 - B_i) \sum_j \Theta^j_i .$$

Since $B_i \neq 0$, it follows that, on any integral manifold, $\Theta^i_j = 0$ for every i and j. The integrability condition now follows using the decomposition (13).

Having shown that there exist quasi-hyperbolic metrics for which this integrability condition holds, [BBG] go on to determine whether or not the system is involutive. Its tableau has characters $s_1 = n(n-1)$, $s_2 = \ldots = s_n = 0$. Integral elements are obtained by setting all forms in I equal to zero, choosing $n(n-1)$ constants $\{A^i_j | i \neq j\}$ and setting

$$\begin{aligned} \tilde{\nu}^i_j &= -\lambda_i \tilde{\omega}^j + \lambda_j \tilde{\omega}^i + A^i_j B_j \tilde{\omega}^j - A^j_i B_i \tilde{\omega}^i \\ \pi_{ij} &= A^j_i B_i \tilde{\omega}^j - A^i_j B_i \tilde{\omega}^i . \end{aligned} \tag{22}$$

Since the space of integral elements is $n(n-1)$ dimensional, the system is involutive and local solutions depend on $n(n-1)$ functions of one variable.

§3. The minimality condition

Now we add the requirement that the image of the isometric embedding be a minimal submanifold. The minimality condition $\sum_{i,j} g^{ij} h^\mu_{ij} = 0$ and (9) imply that

$$\sum_i b_i g^{ii} = 0. \tag{23}$$

Our system for minimal isometric immersions will be I restricted to the submanifold $\Sigma''' \subset \Sigma''$ where (23) holds.

Any set of vectors b_i satisfying (10) have exactly one linear relation among them, and up to multiple this must be

$$\sum_i \frac{b_i}{B_i} = 0. \tag{24}$$

Using (24) and (20), one gets an equivalent minimality condition,

$$\frac{1}{B_i} = \frac{g^{ii}}{\sum_j g^{jj}} . \tag{25}$$

In order to see if any integral elements now exist for I, we will need to see if (22) is compatible with the additional linear relations on the one-forms π_{ij} and $\tilde{\omega}^i_j$ introduced by the restriction to Σ'''.

On Σ'', the relations (10) implied $dB_i = 2B_i \underset{j}{\Sigma} \pi_{ij}$. We already had the relations

$$B_j \pi_{ij} + B_i \pi_{ji} = 0, \qquad i \neq j. \tag{26}$$

We also note that

$$dg^{ij} = -g^{ik}\tilde{\omega}^j_k - g^{jk}\tilde{\omega}^i_k.$$

Now, differentiating (25) gives

$$\sum_j \frac{\pi_{ij}}{B_i} = (\sum_j g^{ij}\tilde{\omega}^i_j - g^{ii}\sum_{j,k} g^{jk}\tilde{\omega}^j_k)/(\underset{j}{\Sigma} g^{jj}). \tag{27}$$

These constitute $n-1$ additional linearly independent relations on the π_{ij}. Substituting (19) and (22) into (27) gives

$$B_i g^{ii} \sum_j (A^j_i \tilde{\omega}^j - A^i_j \tilde{\omega}^i) = \sum_j g^{ij}(A^i_j B_j \tilde{\omega}^j - A^j_i B_i \tilde{\omega}^i + 2\lambda_i \tilde{\omega}^j) - 2\frac{1}{B_i}\sum_{j,k} g^{jk}\lambda_j \tilde{\omega}^k,$$

where for convenience we set $A^i_i = 0$ for all i. For $k \neq i$, taking the coefficient of $\tilde{\omega}^k$ on each side gives

$$B_i g^{ii} A^k_i = g^{ik}(A^i_k B_k + 2\lambda_i) - 2\frac{1}{B_i}\sum_j g^{jk}\lambda_j.$$

(The equations obtained by taking the coefficient of $\tilde{\omega}^i$ will be linearly dependent on the last set.) This equation, together with that obtained by interchanging i and k, gives a pair of linear equations for A^k_i and A^i_k,

$$\begin{aligned} B_i g^{ii} A^k_i - B_k g^{ik} A^i_k &= 2g^{ik}\lambda_i - 2\frac{1}{B_i}\sum_j g^{jk}\lambda_j \\ B_k g^{kk} A^i_k - B_i g^{ik} A^k_i &= 2g^{ik}\lambda_k - 2\frac{1}{B_k}\sum_j g^{ij}\lambda_j. \end{aligned} \tag{28}$$

Since the coefficient matrix on the left has determinant $B_i B_k (g^{ii}g^{kk} - (g^{ik})^2)$, and this is clearly nonzero, these equations determine the coefficients A^i_j uniquely in terms of functions defined on $\tilde{\mathcal{F}}$. We conclude that there is a unique integral n-plane satisfying the independence condition at each point of Σ'''. In fact, we may define a new Pfaffian system on Σ''' by adding more generator 1-forms to I:

$$J = I \oplus \{\tilde{\nu}^i_j + \lambda_i \tilde{\omega}^j - \lambda_j \tilde{\omega}^i - A^i_j B_j \tilde{\omega}^j + A^j_i B_i \tilde{\omega}^i, \pi_{ij} - A^j_i B_i \tilde{\omega}^j + A^i_j B_i \tilde{\omega}^i\}, \tag{29}$$

where the A^i_j are determined by (28). At each point of Σ''', J annihilates our distribution of n-planes. Any integral manifold of I in Σ''' satisfying the independence condition will be an integral manifold of J. At this point we see that solutions depend at most on constants.

Proof of Theorem C. When $(M, \tilde{g})$ is itself a space form of constant sectional curvature k, a trace of the Gauss equations implies (as in the proof of [M], Thm. 2) that $\epsilon \geq k$. If $\epsilon = k$ then M is totally geodesic. If $\epsilon > k$, the Gauss equations take the form (8) with $\widehat{II} = II \oplus \sqrt{\epsilon - k}\,\tilde{g}$. Then Cartan's theorem implies the rank is at least $n-1$.

Now assume the rank is $n-1$; if the codimension exceeds the rank, extra 1-forms are present in the system I, but the 2-forms are unchanged, so we add the same 1-forms as in (29) to obtain system J. When we take $Q = \sqrt{\epsilon - k}\,\tilde{g}$, the forms $\tilde{\mu}^i_j$ are automatically zero, as are the right-hand sides in (28). Now J takes the form

$$J = I \oplus \{\tilde{\nu}^i_j, \pi_{ij}\}.$$

However, because

$$\begin{aligned} d\tilde{\nu}^i_j &= -\tilde{\nu}^i_k \wedge \tilde{\nu}^k_j - \tilde{\omega}^i \wedge \tilde{\omega}^j \\ &\equiv -k\tilde{\omega}^i \wedge \tilde{\omega}^j \mod J, \end{aligned}$$

we see that unless $k = 0$, J satisfies the Frobenius condition nowhere on Σ'''. Uniqueness in the flat case now follows by the argument given at the end of the proof of Theorem D.

More generally, if Q is $\tilde{g}$-parallel, then the forms $\tilde{\mu}^i_j$ and tensor components λ_i must vanish. Once again, $J = I \oplus \{\tilde{\nu}^i_j, \pi_{ij}\}$, and the argument proceeds as above.

Returning to the general case, we assume now that $(M^n, \tilde{g})$ is a quasi-κ curved Riemannian manifold with respect to a positive-definite Q, such that (19) holds.

Any integral n-manifold of J will push down to an integral n-manifold in $\tilde{\mathcal{F}}$ for the system

$$\tilde{J} = \{\tilde{\nu}^i_j + \lambda_i\tilde{\omega}^j - \lambda_j\tilde{\omega}^i - A^i_j B_j\tilde{\omega}^j + A^j_i B_i\tilde{\omega}^i\},$$

where B_i and A^i_j now are determined by by the functions $\tilde{g}^{ij}$ and λ_i on $\tilde{\mathcal{F}}$. So, the half of the Frobenius condition for J that involves $d\tilde{\nu}^i_j$ will only require the vanishing of certain functions on $\tilde{\mathcal{F}}_Q$, and these are intrinsic integrability conditions that depend only on the metric $\tilde{g}$ and on Q. In order to examine the other half, we need to compute $d\pi_{ij}$ modulo I.

Differentiating (12) gives

$$\begin{aligned} &\sum_{j,k} b^\mu_j \tilde{g}^{jk}(b_i \cdot b_k)\tilde{\omega}^j \wedge \tilde{\omega}^k \\ &\equiv \sum_j \left[\left(\sum_k (b^\mu_i - b^\mu_k)\pi_{ik} - \sum_k (b^\mu_j - b^\mu_k)\pi_{jk} \right) \wedge \pi_{ij} + (b^\mu_i - b^\mu_j)d\pi_{ij} \right] \mod I. \end{aligned}$$

Dotting with b_p, $p \neq i$, gives

$$\sum_{j,k} \tilde{g}^{jk}(b_j \cdot b_p)(b_i \cdot b_k)\tilde{\omega}^j \wedge \tilde{\omega}^k$$
$$\equiv \sum_j \left(B_p \pi_{ip} - \sum_k ((b_j \cdot b_p) - (b_k \cdot b_p))\pi_{jk} \right) \wedge \pi_{ij} + B_p d\pi_{ip} \bmod I.$$

Using (16), we see that the value of $d\pi_{ij}$ modulo I can be expressed in terms of the π_{ij}'s themselves and functions and forms defined on $\tilde{\mathcal{F}}_Q$, so these integrability conditions also are intrinsic. In other words, *the Frobenius condition for J can be expressed solely in terms of the vanishing of certain functions on $\tilde{\mathcal{F}}_Q$.*

Proposition. *Given a section $\{\tilde{e}_i\}$ of $\tilde{\mathcal{F}}_Q$ along which the aforementioned functions vanish, there exists a minimal isometric embedding of M into X, unique up to rigid motion, under which $\{\tilde{e}_i\}$ are the principal tangent directions.*

Proof. Existence follows from the Frobenius theorem. Suppose now there are two such embeddings f, F. Fixing a point $p \in M$, we can arrange by rigid motions that $f(p) = F(p) = x$ and $f_*(T_pM) = F_*(T_pM) = T_x$. Now we have

$$g_{ij} =< f_*\tilde{e}_i, f_*\tilde{e}_j >=< F_*\tilde{e}_i, F_*\tilde{e}_j >$$

and it follows that we can arrange, by rotations of X acting on the plane T_x, that $f_*\tilde{e}_i = F_*\tilde{e}_j$. We can also arrange that the principal normal vectors

$$b_i = \sum_\mu b_i^\mu e_\mu \in N_x$$

also coincide. (For, the b_i are n vectors with $b_i \cdot b_j = -1 + \delta_{ij} \sum_k \tilde{g}^{kk}/\tilde{g}^{ii}$, and the set of such vectors is acted on simply transitively by rotations in N_x.) To each of f and F there is associated a "graph", a section of $\Sigma''' \subset \tilde{\mathcal{F}}_Q \times \mathcal{F} \times \mathcal{W}$ which covers the embedding, and which is an integral manifold of J. By using the action of $O(n-1)$ along the Cauchy characteristics of J, i.e. the action by orthogonal substitions among the frame vectors e_μ, we can arrange that the frames associated to f and F coincide at x. Then the vectors b_i^μ must coincide there as well. This means that the integral manifolds of J associated to f and F go through the same point of Σ''' above p, and so the embeddings must coincide everywhere else.

Proof of Theorem D. Again, fix a generic point $p \in M$, and let G be the identity component of the group of isometries of M fixing p. Then, because G fixes the Ricci tensor at p, it also fixes Q. The argument of the above proposition could be used to prove that a minimal isometric embedding of M is unique up to rigid motion if we knew that G acted simply transitively on Q-orthonormal frames $\{\tilde{e}_i\}$. (This is the case, of course, when $Q = \tilde{g}$.) For, if f and F are two such

embeddings, and the associated sections of $\tilde{\mathcal{F}}_Q$ are, at p, in the same orbit of G, then we can arrange by isometries of M that they are the same at p, and (as in the proposition) arrange by rigid motions of X that the embeddings coincide. Hence the embeddings depend, up to rigid motions, only on the value of $\{\tilde{e}_i\}$ at p, modulo the action of G.

§4. Isometric Embeddings with Parallel Mean Curvature

Suppose $(M, \tilde{g})$ is a quasi-κ-curved Riemannian manifold, and M is isometrically embedded in space form $X^{2n-1}(\kappa+1)$. In terms of the usual adapted framing (see §2), the mean curvature vector is

$$H = \sum_{\mu,i,j} e_\mu g^{ij} h^\mu_{ij} = \sum_{\mu,i} e_\mu g^{ii} b^\mu_i.$$

Using (12) and the structure equations of $\mathcal{F}$, the normal component of ∇H is

$$\nabla_N H = \sum_{\mu,i,j} e_\mu \left(g^{ii}(b^\mu_i - b^\mu_j)\pi_{ij} - 2g^{ij} b^\mu_i \omega^i_j \right).$$

Thus, requiring that the mean curvature vector be parallel amounts to requiring that the vector-valued 1-forms

$$\sum_{i,j} \left(g^{ii}(b_i - b_j)\pi_{ij} - 2g^{ij} b_i \omega^i_j \right)$$

vanish, in addition to the forms in the system I defined by (11). Dotting with vector b_k gives the equivalent condition

$$0 = \sum_j \left(\pi_{kj} g^{kk} - \pi_{jk} g^{jj} - 2g^{jk} \tilde{\omega}^k_j \right) + \frac{2}{B_k} \sum_{i,j} g^{ij} \tilde{\omega}^i_j. \tag{30}$$

Thus, such isometric embeddings will arise as integral manifolds of the following Pfaffian system on Σ'':

$$K = I \oplus \{\sum_j \left(\pi_{kj} g^{kk} - \pi_{jk} g^{jj} - 2g^{jk} \tilde{\omega}^k_j \right) + \frac{2}{B_k} \sum_{i,j} g^{ij} \tilde{\omega}^i_j\}.$$

Now we may substitute in (30) the values of π_{ij} and $\tilde{\omega}^i_j = \tilde{\mu}^i_j + \tilde{\nu}^i_j$ on a typical integral element of I, as given by (19) and (22), to get the condition

$$\sum_j A^j_k (B_k g^{kk} + B_j g^{jj}) \tilde{\omega}^j - A^k_j (B_k g^{kk} + B_j g^{jj}) \tilde{\omega}^k - 2g^{jk} (A^k_j B_j \tilde{\omega}^j - A^j_k B_k \tilde{\omega}^k)$$
$$= 4(\textstyle\sum_j g^{jk} \lambda_j \tilde{\omega}^k - \frac{1}{B_k} \sum_{i,j} g^{ij} \lambda_j \tilde{\omega}^i) + 2(g^{kk} - \frac{1}{B_k} \sum_j g^{jj})(\sum_l \lambda_l \tilde{\omega}^l)$$

For $i \neq k$, equating the coefficients of $\tilde{\omega}^i$ on both sides gives

$$
\begin{aligned}
(B_k g^{kk} + B_i g^{ii})A^i_k - 2g^{ik} B_i A^k_i &= 2\lambda_i (g^{kk} - \frac{1}{B_k}\underset{j}{\Sigma} g^{jj}) - \frac{4}{B_k}\underset{j}{\Sigma} g^{ij}\lambda_j \\
-2g^{ik} B_k A^i_k + (B_k g^{kk} + B_i g^{ii})A^k_i &= 2\lambda_k (g^{ii} - \frac{1}{B_i}\underset{j}{\Sigma} g^{jj}) - \frac{4}{B_i}\underset{j}{\Sigma} g^{kj}\lambda_j
\end{aligned}
$$

(The second equation comes from the first by interchanging i and k.) This gives two linear equations for A^i_k and A^k_i, and the determinant of the coefficient matrix is

$$
(B_k g^{kk} + B_i g^{ii})^2 - 4B_i B_k (g^{ik})^2 = (B_k g^{kk} - B_i g^{ii})^2 + 4B_i B_k (g^{ii} g^{kk} - (g^{ik})^2) > 0.
$$

So, as happened with the minimality condition in §3, at each point of Σ'' there is a unique integral n-plane for K. Now the arguments in the proofs of Theorems C and D apply as before. (Note, however, that tracing the Gauss equations, as we do in the proof of Theorem C, does not yield any useful inequalities for the curvature of M and X in this case.)

§5. Construction of quasi-κ-curved metrics

Let $e_0, \ldots, e_{n+1}$ be an orthornormal frame of $\mathbb{L}^{n+2}$ with e_0 a timelike direction. On the orthonormal frame bundle $\mathcal{F}_{\mathbb{L}}$ of $\mathbb{L}^{n+2}$, we have the Maurer-Cartan form

$$
\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^0 & 0 & \omega^0_i & \omega^0_{n+1} \\ \omega^j & \omega^j_0 & \omega^i_j & \omega^j_{n+1} \\ \omega^{n+1} & \omega^{n+1}_0 & \omega^{n+1}_j & 0 \end{pmatrix}
$$

where $\omega^i_j = -\omega^j_i, \omega^0_i = \omega^i_0, \omega^0_{n+1} = \omega^{n+1}_0, \omega^j_{n+1} = -\omega^{n+1}_j$.

Assume $M^n \subset \mathbb{L}^{n+2}$ is spacelike. Let $\mathcal{F}^1_{\mathbb{L}}$ denote the subbundle of the restriction of $\mathcal{F}_{\mathbb{L}}$ to M consisting of frames such that at each point $x \in M$, $T_xM = \{e_1, \ldots, e_n\}$. On $\mathcal{F}^1_{\mathbb{L}}$, one has

$$
\omega^0_i = q_{ij}\omega^j, \;\; \omega^{n+1}_i = h_{ij}\omega^j
$$

for some functions q_{ij}, h_{ij} symmetric in their lower indices. The curvature tensor of M is

$$
R = \gamma(h,h) - \gamma(q,q)
$$

To construct quasi-κ-curved metrics, we want

$$
h = \sqrt{\kappa+1}g
$$

where g is the induced metric on M. Since we are working with g-orthonormal frames, we need $h_{ij} = \sqrt{\kappa+1}\delta_{ij}$.

In what follows, we will repeatedly differentiate the above conditions, using the Maurer-Cartan structure equations. Since we will be using the results of [Lan], we will generally use formulae and notation from there, except that the indices i, j, k will index the tangent directions, instead of α, β, γ. μ, ν will index normal directions, in this case just 0 and $n+1$. In particular, r^{μ}_{ijk} will denote the coefficients of F_3, which in our situation is a tensor on M, namely ∇II. Similarly, $F_k = \nabla^{k-2} II$.

The coefficients of F_3, F_4, F_5 are obtained by differentiating $\omega^{\mu}_j - q^{\mu}_{jk}\omega^k$. They are given as follows:

$$\begin{aligned}
r^{\mu}_{ijk}\omega^k &= -dq^{\mu}_{ij} - q^{\nu}_{ij}\omega^{\mu}_{\nu} + q^{\mu}_{il}\omega^l_j + q^{\mu}_{jl}\omega^l_i \\
r^{\mu}_{ijkl}\omega^l &= -dr^{\mu}_{ijk} - r^{\nu}_{ijk}\omega^{\mu}_{\nu} + \mathfrak{S}_{ijk} r^{\mu}_{ijl}\omega^l_k - \mathfrak{S}_{ijk} q^{\mu}_{il} q^{\nu}_{jk}\omega^l_{\nu} \\
r^{\mu}_{ijklm}\omega^m &= -dr^{\mu}_{ijkl} - r^{\nu}_{ijkl}\omega^{\mu}_{\nu} + \mathfrak{S}_{ijkl} r^{\mu}_{ijkm}\omega^m_l - \mathfrak{S}_{ijkl}(r^{\mu}_{ijm} q^{\nu}_{kl} + q^{\mu}_{im} r^{\mu}_{jkl})\omega^m_{\nu},
\end{aligned}$$

where $\mathfrak{S}_{ijk}$ denotes a cyclic sum over i, j, k and $\mathfrak{S}\mathfrak{S}_{ijkl}$ denotes summing to symmetrize the expression over i, j, k, l.

Finally, because results in [Lan] are phrased in terms of submanifolds of complex projective space, we will consider $\mathbb{L}^{n+2} \subset \mathbb{RP}^{n+2} \subset \mathbb{CP}^{n+2}$ and $\mathcal{F}_{\mathbb{L}} \subset \mathcal{F}_{\mathbb{CP}^{n+2}}$.

Proof of Lemma. To indicate how the proof applies to more general signatures, we will let $\epsilon_\mu = -< e_\mu, e_\mu >= \pm 1$; then $\omega^{\mu}_j = \epsilon_\mu \omega^j_\mu$.

Using ([Lan], 2.15), or by differentiating the equation $\omega^{n+1}_j - \sqrt{\kappa+1}\omega^j = 0$, we obtain

$$r^{n+1}_{ijk}\omega^k = -q_{ij}\omega^{n+1}_0 \qquad \text{and} \qquad \omega^{n+1}_0 \wedge \epsilon_0 q_{ij}\omega^j = 0 \;\; \forall i.$$

Since Q is nondegenerate, this implies that

$$r^{n+1}_{ijk} = 0 \text{ and } \omega^{n+1}_0 = 0$$

which proves the assertion $\sigma \lrcorner \nabla II = 0$. (Note that the normal bundle splits into parallel sub-bundles spanned by σ and $\sigma^\perp$.) Now

$$\begin{aligned}
r^{n+1}_{ijkl}\omega^l &= \mathfrak{S}_{ijk} q^{n+1}_{im} q^{\mu}_{jk}\omega^m_{\mu} \\
&= \epsilon_\mu \sqrt{\kappa+1}\mathfrak{S}_{ijk} q^{\mu}_{im} q^{\mu}_{jk}\omega^m
\end{aligned}$$

so that

$$r^{n+1}_{ijkl} = \epsilon_\mu \sqrt{\kappa+1}\mathfrak{S}\mathfrak{S}_{ijkl} q^{\mu}_{il} q^{\mu}_{jk},$$

which implies that ([Lan], 4.13) holds with $b^{n+1}_{\mu\nu} = \delta_{\mu\nu}\epsilon_\mu\sqrt{\kappa+1}$.

Now to compute coefficients of F_5:

$$\begin{aligned}
r^{n+1}_{ijklm}\omega^m &= -dr^{n+1}_{ijkl} + \mathfrak{S}_{ijkl} r^{n+1}_{ijkm}\omega^m_l - \mathfrak{S}_{ijkl} q^{n+1}_{im} r^{\mu}_{jkl}\omega^m_{\mu} \\
&= \epsilon_\mu \sqrt{\kappa+1}\mathfrak{S}\mathfrak{S}_{ijkl} d(q^{\mu}_{ij} q^{\mu}_{kl}) - \mathfrak{S}_{ijk}(\mathfrak{S}\mathfrak{S}_{ijkm} q^{\mu}_{ij} k q^{\mu}_{km})\omega^m_l) - \epsilon_\mu \mathfrak{S}_{ijkl} r^{\mu}_{jkl} q^{\mu}_{im}\omega^m
\end{aligned}$$

When we substitute

$$dq^{\mu}_{ij} = -r^{\mu}_{ijm}\omega^m + q^{\mu}_{im}\omega^m_j + q^{\mu}_{jm}\omega^m_i - q^{\nu}_{ij}\omega^{\mu}_{\nu},$$

everything in $r^{n+1}_{ijklm}\omega^m$ cancels except a term of the form $\epsilon_\mu r^{\mu}_{ijk}q^{\mu}_{lm}\omega^m$, which implies that ([Lan], 4.16) holds as well. This implies that at each point there is a quadric, whose tangent space is ${e_{n+1}}^{\perp}$, osculating to order five. Since we are in codimension two, Q is nonzero, and there is at least one quadric of rank at least three, we know there are no linear syzygies among the quadrics in II. We thus conclude by ([Lan], 4.20) that the quadric actually contains M. Computing at the point $(0,\dots,0)$, following ([Lan], 4.18) we find that the equation of the quadric is:

$$x^{n+1} - \sqrt{k+1}((x^1)^2 + \cdots + (x^n)^2 - (x^0)^2) + (x^{n+1})^2) = 0$$

and completing the square yields (3).

We now examine the quasi-κ-curved metrics satisfying the integrability conditions; i.e., we want metrics such that

$$e_0 \lrcorner \nabla II = L \circ (e_0 \lrcorner II). \tag{31}$$

This implies that at each point of M there is a quadric hypersurface having tangent plane ${e_0}^{\perp}$ osculating to order three. Having a quadric osculate to order three at a general point is not enough to imply containment; but our additional conditions will imply containment.

We will generally follow [Lan] in the remainder of this section with one exception: in what follows 0 will denote a normal index, and what is denoted x^0 in [Lan] should be taken to be a homogeneous coordinate which we set equal to 1 in restricting to an affine space.

What follows is actually a result in projective geometry; only one needs the hypotheses that one has two sections σ and τ of the normal bundle NM which have the property that $\sigma \lrcorner F_3 = 0$ and $\tau \lrcorner F_3 = L \circ (\tau \lrcorner II)$. Then one can restrict to $\sigma \lrcorner II$-orthonormal frames in TM and to frames in NM where σ and τ form pointwise a Minkowski orthonormal basis, and then restrict further to an affine open subset. For simplicity, we will work in Lorentz space.

In frames, our hypothesis is that

$$r^0_{ijk} = \mathfrak{S}_{ijk}\lambda_i q^0_{jk} \tag{32}$$

i.e. that

$$-dq^0_{ij} + q^0_{il}\omega^l_j + q^0_{jl}\omega^l_i = \sum_k(\lambda_i q^0_{jk} + \lambda_j q^0_{ik} + \lambda_k q^0_{ij})\omega^k \tag{33}$$

Computing the coefficients of F_4 (i.e. differentiating (32)) we obtain

$$r^0_{ijkl}\omega^l = -dr^0_{ijk} + \mathfrak{S}_{ijk} r^0_{ijm}\omega^m_k - \mathfrak{S}_{ijk} q^0_{im} q^0_{jk}\omega^m_0 - \mathfrak{S}_{ijk} q^0_{im} q^{n+1}_{jk}\omega^m_{n+1}$$

(34)

$$= \mathfrak{S}_{ijk}\left\{-q^0_{jk}d\lambda_i + \lambda_l q^0_{ij}\omega^l_k + [\lambda_i(\mathfrak{S}_{jkl}\lambda_j q^0_{kl}) + (\kappa+1)q^0_{il}\delta_{jk} - \sum_m q^0_{im}q^0_{ml}q^0_{jk}]\omega^l\right\}$$

Note that (34) includes a term LF_3, which is necessary for the Monge system, a term of potential torsion (obstruction to integrability), and a term of Cauchy characteristics.

To simplify computations, at this point we restrict to frames where q^0 is diagonal and write q_i for q^0_{ii}. Then (33) becomes

$$(q_i - q_j)\omega^i_j = \lambda_i q_j \omega^j + \lambda_j q_i \omega^i \tag{35}$$

$$d(log(q_i)) = -\sum_k \lambda_k \omega^k - 2\lambda_i\omega^i. \tag{36}$$

If $q_i \neq q_j$ we get

$$\omega^i_j = \frac{\lambda_i q_j}{q_i - q_j}\omega^j + \frac{\lambda_j q_i}{q_i - q_j}\omega^i \tag{37}$$

If none of i, j, k are equal, (34) simplifies to

$$r^0_{ijkl}\omega^l = 2q_k\lambda_i\lambda_j\omega^k + 2q_j\lambda_k\lambda_i\omega^j + 2q_i\lambda_j\lambda_k\omega^i \tag{38}$$

Note that this is the fourth order Monge condition ([Lan], 4.17) for these indices.

For simplicity, we first assume that the eigenvalues q_i of Q are distinct on an open set in M. (38) implies that

$$\begin{aligned} r^0_{ijkl} &= 0 && \text{if } ijkl \text{ are all distinct,} \\ r^0_{iijk} &= 2\lambda_j\lambda_k q_i && \text{if } ijk \text{ are all distinct.} \end{aligned} \tag{39}$$

On the other hand, when $i \neq k$, from (34) we get

$$\begin{aligned} r^0_{iikl}\omega^l = &-q_i d\lambda_k + (4\lambda_i\lambda_k q_i)\omega^i \\ &+ [2\lambda_i^2 q_k + \lambda_k^2 q_i + (\kappa+1)q_k - q_k^2 q_i + q_i q_k(\sum_{l\neq k}\frac{\lambda_l^2}{q_l - q_k})]\omega^k \\ &+ \sum_{l\neq k}[\lambda_l\lambda_k q_i(\frac{q_l}{q_l - q_k} + 1)]\omega^l \end{aligned} \tag{40}$$

The symmetry of r^0_{ijkl} in the lower indicies places restrictions on $d\lambda_i$. If we let $d\lambda_k = \lambda_{kl}\omega^l$, then (40) implies, for i, k, l distinct,

(41)

$$r^0_{iikl} = -q_i\lambda_{kl} + \lambda_k\lambda_l q_i(\frac{q_l}{q_l - q_k} + 1)$$

(42)

$$r^0_{iikk} = q_i\left(-\lambda_{kk} + \lambda_k^2 - q_k^2 + q_k(\sum_{l\neq k}\frac{\lambda_l^2}{q_l - q_k})\right) + (\kappa+1)q_k + 2\lambda_i^2 q_k$$

Combining (39) and (41) we obtain

$$\lambda_{kl} = \frac{\lambda_k \lambda_l q_k}{q_l - q_k}, \quad k \neq l.$$

Using the symmetry $r^0_{iikk} = r^0_{kkii}$, we obtain

$$q_i(\lambda_{kk} + \lambda_k^2 + (\kappa+1) + q_k^2 - q_k \sum_{l\neq k} \frac{\lambda_l^2}{q_l - q_k}) = q_k(\lambda_{ii} + \lambda_i^2 + (\kappa+1) + q_i^2 - q_i \sum_{l\neq i} \frac{\lambda_l^2}{q_l - q_i}).$$

Hence

$$\frac{1}{q_k}(\lambda_{kk} + \lambda_k^2 + (\kappa+1) + q_k^2 - q_k \sum_{l\neq k} \frac{\lambda_l^2}{q_l - q_k}) \tag{43}$$

is independent of k; call this quantity $-b^0_{00}$ and set $b^0_{0,n+1} = \kappa + 1$. Then

$$r^0_{iikk} = 2\lambda_i^2 q_k + 2\lambda_k^2 q_i + b^0_{00} q_i q_k + b^0_{0,n+1}(q_i + q_k),$$

and thus the remaining fourth order Monge conditions in the e_0 direction hold as well:

$$F^0_4 = LF^0_3 + b^0_{00} F^0_2 F^0_2 + b^0_{0,n+1} F^0_2 F^{n+1}_2. \tag{44}$$

Differentiating again, one sees that the fifth order Monge condition ([Lan],4.18) holds as well. Setting $b = b^0_{00}$ and using ([Lan], 4.18) one gets (4).

Now assume that some eigenvalues of Q have multiplicity one and some have multiplicity greater than one on an open set in M; let ξ, η index the former and α, β index the latter. Then (35) implies $\lambda_\alpha = 0$ and

$$\omega^\xi_\alpha = \frac{\lambda_\xi q_\alpha}{q_\xi - q_\alpha} \omega^\alpha.$$

Differentiating this gives the analogous results:

$$\lambda_{\xi\alpha} = 0, \qquad \lambda_{\xi\eta} = \frac{\lambda_\xi \lambda_\eta q_\xi}{q_\eta - q_\xi}, \quad \xi \neq \eta,$$

and

$$\frac{1}{q_\xi}(\lambda_{\xi\xi} + \lambda_\xi^2 + (\kappa+1) + q_\xi^2 - q_\xi \sum_{\eta\neq\xi} \frac{\lambda_\eta^2}{q_\eta - q_\xi}) = \frac{1}{q_\alpha}((\kappa+1) + q_\alpha^2 - q_\alpha \sum_\eta \frac{\lambda_\eta^2}{q_\eta - q_\alpha}) = -b^0_{00},$$

independent of α and ξ. Now the fourth-order Monge condition (44) holds without change, as does the fifth-order condition.

The case where the eigenvalues of Q all have multiplicity greater than one is simple: (35) implies that $\lambda_i = 0$ for all i, hence the q_i are constant, and M is a product of space forms given by the intersection of (3) and (5).

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